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N-flips in even triangulations on surfaces

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ABSTRACT

In this paper, we show that any two even triangulations on the same closed surface with the same and sufficiently large number of vertices can be transformed into each other by a sequence of two specifically defined deformations called an *N*-flip and a P_2 -flip, up to homeomorphism, if they have the same homological structure.

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1. Introduction

A triangulation G of a closed surface F^2 is a simple graph embedded on F^2 so that each face is triangular. A triangulation G is said to be *even* if each vertex of G has even degree. For a vertex v of G , the *link* of v is the boundary cycle of the 2-cell region formed by the faces of G incident to v .

Suppose that an even triangulation G has a hexagonal region $v_1v_2v_3v_4v_5v_6$ with diagonals v_1v_3 , v_3v_6 and v_4v_6 and no inner vertices. The *N-flip* of the path $v_1v_3v_6v_4$ is to replace the diagonals v_1v_3 , v_3v_6 and v_4v_6 with v_1v_5 , v_2v_5 and v_2v_4 in the hexagonal region, as shown in the left-hand side of Fig. 1. If the resulting graph is not simple, then we do not apply it.

Let G be an even triangulation on a closed surface and let v be a vertex of G with link $v_1 \cdots v_k$. Put two vertices x and y on vv_1 and join them to v_2 and v_k , and let G' be the resulting graph. The P_2 -flip of $\{x, y\}$ in G' is to move the inserted vertices x and y to the edge vv_2 to join them to v_1 and v_3 , as shown in the right-hand side of Fig. 1. Similarly to the above, if the resulting graph is not simple, then we do not apply the operation.

It is easy to see that both of *N*- and P_2 -flips transform an even triangulation into an even triangulation. The following theorem is known [15].

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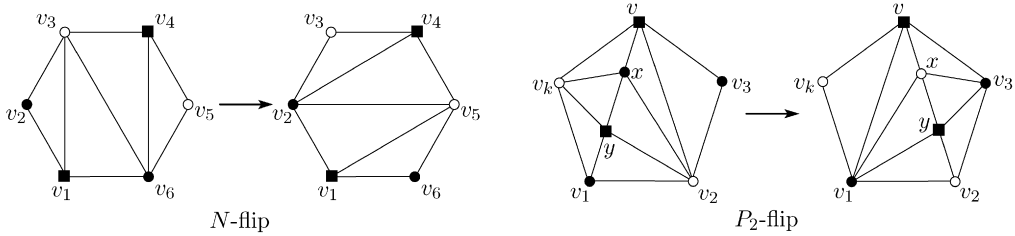


Fig. 1. Two local deformations.

Theorem 1 (Nakamoto, Sakuma and Suzuki). *Any two even triangulations on the sphere with the same number of vertices can be transformed into each other by a sequence of N - and P_2 -flips.*

A k -coloring of a graph G is a map $c : V(G) \rightarrow \{1, \dots, k\}$ such that for any $xy \in E(G)$, $c(x) \neq c(y)$. We say that G is k -colorable if G admits a k -coloring, and that G is k -chromatic if G is k -colorable but not $(k-1)$ -colorable. It is well known that every even triangulation on the sphere is 3-chromatic.

In this paper, we would like to extend Theorem 1 to all closed surfaces. However, it is known that each non-spherical surface admits a non-3-colorable even triangulation. As shown in Fig. 1, the two operations preserve the 3-chromaticity of a given even triangulation, and hence a 3-chromatic one and a non-3-chromatic one cannot be transformed into each other by the two transformations even if they have the same and sufficiently large number of vertices. For the projective plane, it has been shown in [16] that any two even triangulations G and G' can be transformed into each other by N - and P_2 -flips, if both or neither of them are 3-chromatic. In this paper, we would like to consider what conditions two even triangulations on all surfaces need to transform into each other by N - and P_2 -flips.

This research on even triangulations has been motivated by the earlier results on diagonal flips in triangulations [3,7,17–20] and quadrangulations [9–11]. (See the survey papers [11,19] for triangulations and quadrangulations, respectively.)

2. Main theorems and monodromies of even triangulations

In this section, we describe our main theorems for even triangulations on all surfaces after introducing an important invariant for even triangulations called “monodromies.”

Let G be an even triangulation of a closed surface F^2 and let f be a face of G (called an *initial face*) with three corners v_1, v_2, v_3 . Let $W = f_0 f_1 \cdots f_k$ be a sequence of faces f_i of G , called a *face walk*, such that $f_0 = f$, and f_i and f_{i+1} share an edge, for $i = 0, 1, \dots, k-1$. (Let W^* denote the walk of the surface dual G^* of G corresponding to W .) For $i = 1, \dots, k$, we recursively define a bijection $c_i : V(f_0) \rightarrow V(f_i)$ so that

- (i) for $i = 0$, $c_0 : V(f_0) \rightarrow V(f_0)$ is the identity, and
- (ii) for $i \geq 1$, c_{i-1} and c_i coincide on the two vertices in $V(f_{i-1}) \cap V(f_i)$.

Then we get a bijection $c_k : V(f_0) \rightarrow V(f_k)$, which is called the W -bijection. (See Fig. 2.) Suppose that $k \geq 2$ and $f_0 = f_k$, that is, W is a *closed face walk*. Then we can regard $c_k : V(f) \rightarrow V(f)$ as a permutation on $V(f) = \{v_1, v_2, v_3\}$, called the W -permutation. By the indices 1, 2, 3 of v_i in $V(f)$, we can define $\sigma_G(W, f) \in S_3$, where S_3 is the symmetric group of degree 3.

It is easy to see that if we let W' be another closed face walk passing through f such that W^* and W'^* are homotopic on F^2 , then we have $\sigma_G(W, f) = \sigma_G(W', f)$. (This follows from the fact that each vertex v of G has even degree and hence the wheel consisting of v and its link is 3-colorable.) Moreover, we can see that if W^* bounds a 2-cell on F^2 , then we have $\sigma_G(W, f) = \text{id}$. (Observe that W is homotopic to a closed face walk W' passing through all faces incident to v_1 exactly once, where $v_1 \in V(f)$. Clearly, we have $\sigma_G(W', f) = \text{id}$, since the wheel consisting of v and its link is 3-colorable. Hence we have $\sigma_G(W, f) = \text{id}$.) Therefore we can regard $\sigma_G(W, f)$ as a homomorphism

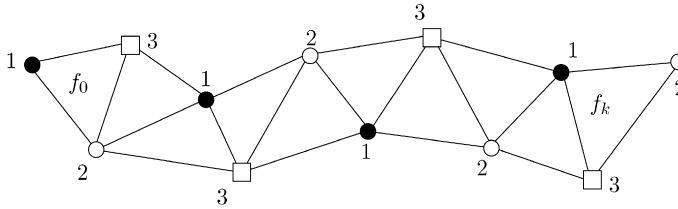
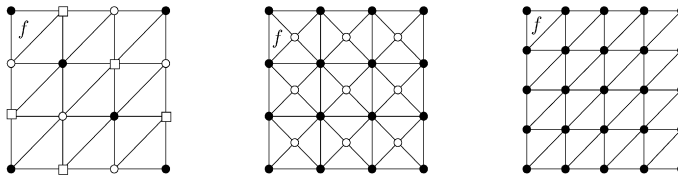
Fig. 2. W -bijection.

Fig. 3. Even triangulations on the torus with non-congruent monodromies.

from the fundamental group $\pi_1(F^2, p)$ to S_3 , where p is a base point on F^2 corresponding to f . We denote it by $\sigma_{G,f} : \pi_1(F^2, p) \rightarrow S_3$, and call it a *monodromy* of G .

Let G and G' be two even triangulations on F^2 . We say that G and G' have the *same* monodromies if there exists a choice of initial faces f with $V(f) = \{v_1, v_2, v_3\}$ and f' with $V(f') = \{v'_1, v'_2, v'_3\}$ of G and G' respectively such that $\sigma_{G,f} = \sigma_{G',f'}$. Our argument for the monodromies does not depend on the choice of initial faces, and so we denote the monodromy of an even triangulation G simply by σ_G .

Observe that an N -flip and a P_2 -flip preserve the monodromy of even triangulations, since these deformations do not change its local 3-coloring, as shown in Fig. 1. That is, if two even triangulations G and G' can be transformed by N - and P_2 -flips, then $\sigma_G = \sigma_{G'}$.

For two even triangulations G and G' on F^2 , σ_G and $\sigma_{G'}$ are said to be *congruent* and denoted by $\sigma_G \equiv \sigma_{G'}$, if G and $h(G')$ have the same monodromy for some homeomorphism $h : F^2 \rightarrow F^2$. Then, if two even triangulations G and G' can be transformed by the two deformations, up to homeomorphism, then we have $\sigma_G \equiv \sigma_{G'}$.

Our main theorem is as follows.

Theorem 2. For any closed surface F^2 , there exists an integer N such that any two even triangulations G and G' on F^2 with $|V(G)| = |V(G')| \geq N$ can be transformed into each other by a sequence of N - and P_2 -flips, up to homeomorphism, if and only if $\sigma_G \equiv \sigma_{G'}$.

Let G be a 3-chromatic even triangulation on a surface F^2 . Fixing any face f of G with three corners v_1, v_2, v_3 , we take any closed face walk W starting and ending at f . Since G is 3-colorable, each v_i is mapped to itself by the W -permutation. Therefore, the monodromy σ_G is *trivial*, that is, for any element $l \in \pi_1(F^2, p)$, we have $\sigma_{G,f}(l) = \text{id}$, where p is a base point on F^2 corresponding to f . The left-hand side of Fig. 3 shows a 3-chromatic even triangulation on the torus, where we should identify two vertical segments and two horizontal ones of the rectangle, respectively.

In this case, $V(G)$ can uniquely be decomposed into three independent sets $V_r(G) \cup V_b(G) \cup V_y(G)$, where these classes are referred as *red*, *blue* and *yellow* vertices of G , respectively. Such a decomposition of $V(G)$ is called the *tripartition* of $V(G)$. The set $\{|V_r(G)|, |V_b(G)|, |V_y(G)|\}$ is called the *tripartition size* of G . Observe that an N -flip preserves the tripartition of $V(G)$, but a P_2 -flip does not, as shown in Fig. 1. Therefore, two 3-chromatic triangulations with the same number of vertices but distinct tripartition size cannot be transformed only by N -flips.

The following is the theorem without P_2 -flips for 3-chromatic triangulations. We say that two even triangulations G and G' are N -equivalent and denoted by $G \approx_N G'$ if they can be transformed into each other only by N -flips, up to homeomorphism.

Theorem 3. For any closed surface F^2 , there exists a set $\{N_r, N_b, N_y\}$ of positive integers such that any two 3-chromatic triangulations G and G' on F^2 are N -equivalent, up to homeomorphism, if they have the same tripartition size $\{N_r + m, N_b + n, N_y + l\}$ for some non-negative integers m, n and l satisfying triangle inequality.

Let D be a non-bipartite even embedding on a closed surface F^2 , that is, an embedding on F^2 such that each face is bounded by a cycle of even length. The *face subdivision* of D , denoted by $S(D)$, is the embedding obtained from D by adding a new single vertex into each face of D and joining it to all vertices on the corresponding boundary cycle. The set of the vertices added to D is called the *color factor* of $S(D)$. Clearly, $S(D)$ is an even triangulation on F^2 . An even triangulation obtained from a non-bipartite even embedding by a face subdivision is called a *proper face subdivision*. (Note that if G is a 3-colorable triangulation on a closed surface F^2 , then each of the three color class is a color factor of G , but the embedding obtained from G by removing a color factor is a *bipartite* even embedding.)

Let G be an even triangulation and let f be a face of G with three corners v_1, v_2, v_3 . The color factor of G can be defined to be the set S of vertices s such that exactly one of v_1, v_2, v_3 is mapped to s by the W -bijection for any face walk W from f to a face incident to s . Let G be a proper face subdivision obtained from a non-bipartite even embedding H , in which we color the vertices of H by black and the color factor by white. (See the center of Fig. 3.) It is easy to see that for any face walk W of G , a white vertex of the initial face must be mapped to a white vertex of the terminal face by the W -bijection. On the other hand, for a closed face walk W through a face f , the W -permutation exchanges two black vertices of f if and only if W_H has odd length, where W_H is the closed walk of H obtained from W by removing all white vertices. Since each edge of H is contained in a closed walk of odd length in H (since H is non-bipartite), a black vertex of G cannot be contained in a color factor of G . Hence G has a unique color factor, that is, the set of all white vertices. We denote the color factor of G by $U(G)$ throughout the paper.

Clearly, an N -flip does not change the number of white vertices of G . The following is a theorem for proper face subdivisions only by N -flips.

Theorem 4. For any closed surface F^2 , there exists a set $\{N, N_U\}$ of positive integers such that any two proper face subdivisions G and G' on F^2 with $|V(G)| = |V(G')| \geq N$ and $|U(G)| = |U(G')| \geq N_U$ are N -equivalent, up to homeomorphism, if and only if they have congruent monodromies.

Finally we show the theorem for even triangulations which are neither 3-chromatic nor proper face subdivisions. Such even triangulations are said to be *generous*. (See the right-hand side of Fig. 3.)

Theorem 5. For any closed surface F^2 , there exists an integer N such that any two generous even triangulations G and G' on F^2 with $|V(G)| = |V(G')| \geq N$ are N -equivalent, up to homeomorphism, if and only if $\sigma_G \equiv \sigma_{G'}$.

Clearly, if G is 3-colorable, a P_2 -flip changes its tripartition size. Moreover, if the graph is a proper face subdivision, the operation can change the order of its color factor. Theorem 5 claims that a P_2 -flip is not needed for generous ones.

We prove only Theorem 3 to avoid a repetition of similar arguments. Then suitable modifications will prove Theorems 4 and 5. Finally introducing a P_2 -flip, we prove Theorem 2.

In order to prove our main theorem, we show in Section 4 that for any surface F^2 , there exist only finitely many minimal even triangulations on F^2 with respect to several reductions. The finiteness of minimal triangulations with respect to edge contractions is well known and was proved in many ways [1,4,5,14]. Moreover, a similar fact for quadrangulations is also known. However, the finiteness we proved for even triangulations cannot be immediately obtained from those of triangulations and quadrangulations.

3. Moving vertices of degree 4 and triangles

Let G be an even triangulation and let $e (= rb)$ be an edge of G shared by two faces rbv_1 and rbv_2 . Subdivide e by two vertices u and v to form a path $ruvb$, and add edges vy_i, uy_i for $i = 1, 2$, as

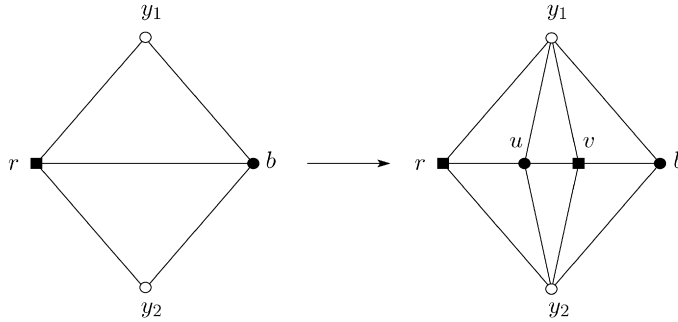


Fig. 4. 2-subdivision of rb .

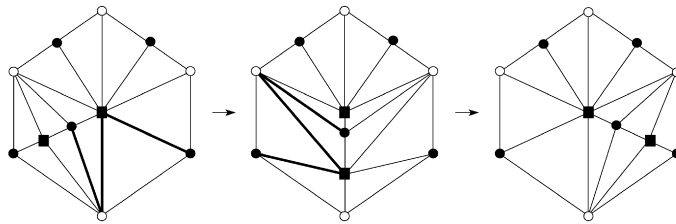


Fig. 5. Moving a 2-subdividing pair of vertices.

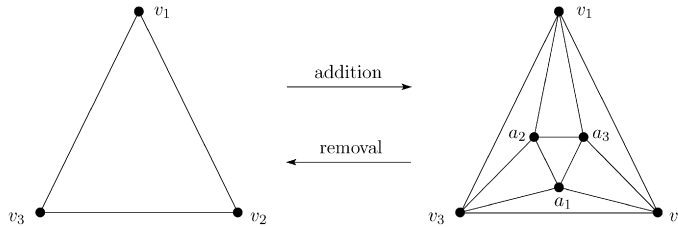


Fig. 6. Addition and removal of an octahedron.

shown in Fig. 4. We call this operation a *2-subdivision* of e . In the resulting graph, $\{u, v\}$ is called a *2-subdividing pair*. If G is 3-chromatic and $r \in V_r(G)$, $b \in V_b(G)$ and $y_1, y_2 \in V_y(G)$, then we have $u \in V_b(G)$ and $v \in V_r(G)$, clearly.

See Fig. 5. We can move a 2-subdividing pair to an alternate edge by exactly two N -flips. This fact suggests the following three lemmas. In fact, the 3-chromatic case was first given in [15]. We say that an edge with red and blue endpoints is an *rb-edge*. Similarly, we can define a *by-edge* and an *ry-edge* as one with blue and yellow endpoints and one with red and yellow endpoints, respectively.

Lemma 6. (See [15].) *Let K be a 3-chromatic triangulation on any surface F^2 , and let e and e' be any two rb -edges of K . Let G (respectively G') be the even triangulation on F^2 obtained from K by a 2-subdivision of e (respectively e'). Then G and G' are N -equivalent.*

Let G be an even triangulation on a closed surface F^2 and let f be a face of G bounded by $v_1v_2v_3$. The *addition of an octahedron* to f is to put a triangle $a_1a_2a_3$ to add edges a_iv_j, a_iv_k for all distinct $i, j, k \in \{1, 2, 3\}$. Its inverse operation is called the *removal of an octahedron*. See Fig. 6.

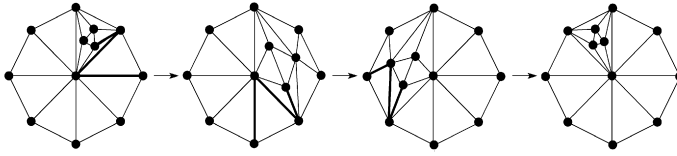
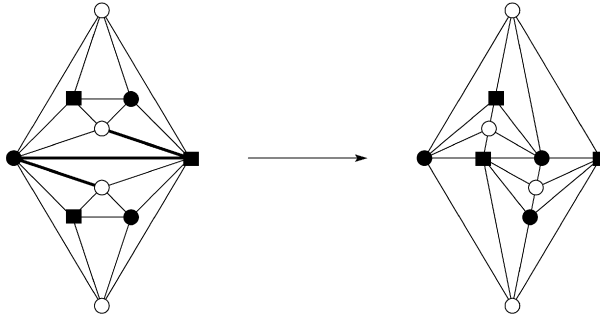


Fig. 7. Moving a triangle.

Fig. 8. N -flip transforming $K + \Delta(2)$ into $K + rb(1) + by(1) + ry(1)$.

Lemma 7. Let K be an even triangulation on any surface F^2 , and let f and f' be any two faces of G . Let G (respectively G') be the even triangulation on F^2 obtained from K by addition of an octahedron to f (respectively f'). Then G and G' are N -equivalent.

Proof. It suffices to prove that an octahedron added to f can be moved to a neighboring face by N -flips. See Fig. 7, in which we can see that the simpleness of the graphs are always preserved. \square

Let G be a 3-chromatic triangulation on a closed surface F^2 . Let $G + rb(p) + by(q) + ry(r)$ denote a 3-chromatic triangulation on F^2 obtained from G by adding p , q and r 2-subdividing pairs to an rb -edge, a by -edge and an ry -edge arbitrarily chosen in G , respectively. This has precisely $|V_R(G)| + p + r$ red vertices, $|V_B(G)| + p + q$ blue vertices and $|V_Y(G)| + q + r$ yellow vertices, and hence has $|V(G)| + 2p + 2q + 2r$ vertices in total. This expresses various even triangulations, depending on the choice of the edges to be 2-subdivided, but Lemma 6 guarantees that they are all N -equivalent, and hence $G + rb(p) + by(q) + ry(r)$ expresses a unique even triangulation, up to N -equivalence. Similarly, let $G + \Delta(m)$ denote an even triangulation obtained from G by the addition of m octahedra. Note that $G + \Delta(m)$ has $|V(G)| + 3m$ vertices. By Lemma 7, $G + \Delta(m)$ expresses a unique even triangulation, up to N -equivalence.

The following lemma connects the two expressions $G + rb(p) + by(q) + ry(r)$ and $G + \Delta(m)$.

Lemma 8. Let G be a 3-chromatic triangulation on a closed surface F^2 and let m be any non-negative integer. Then $G + rb(m) + by(m) + ry(m)$ and $G + \Delta(2m)$ are N -equivalent.

Proof. Since 2-subdividing pairs and octahedra added can be moved to anywhere, by Lemmas 6 and 7, we have only to prove that $K + rb(1) + by(1) + ry(1) \approx_N K + \Delta(2)$. See Fig. 8, which shows that $K + \Delta(2)$ is transformed into $K + rb(1) + by(1) + ry(1)$ by one N -flip. \square

4. Reductions in even triangulations

Let G be an even triangulation on a closed surface F^2 . Let v be a vertex of degree 4 or 6 whose link is $x_0x_1 \cdots x_{2k-1}$, where $k \in \{2, 3\}$. The contraction of v at $\{x_i, x_{i+k}\}$ is to remove v and identify x_{i+i}

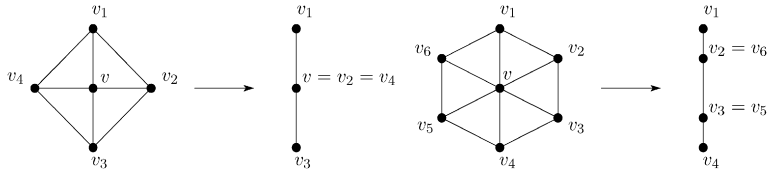


Fig. 9. The contraction of v at $\{v_1, \cdot\}$.

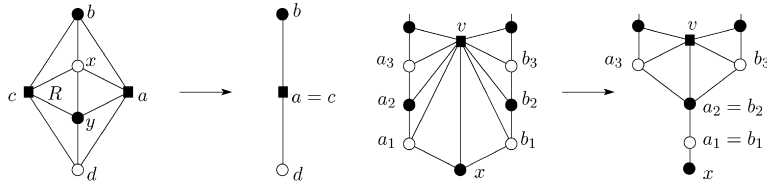


Fig. 10. The 2-elimination and edge elimination.

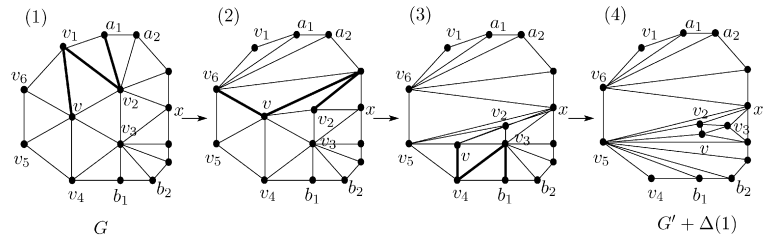


Fig. 11. N -flips making an octahedron.

and x_{l-i} for $i = 1, \dots, k-1$, where the subscripts are taken modulo $2k$. (See Fig. 9.) We say that G is *contractible* at v if for some l , the contraction of v at $\{x_l, x_{l+k}\}$ results in a simple graph.

Suppose that G has a quadrangular region R bounded by a 4-cycle $abcd$ containing only two vertices x and y with $N(x) = \{a, b, c, y\}$ and $N(y) = \{a, c, d, x\}$. The 2-elimination of xy is to delete x and y , identify a and c , and replace two pairs of multiple edges with two single edges, respectively, as shown in the left-hand side of Fig. 10. This operation must preserve the simpleness of the graphs.

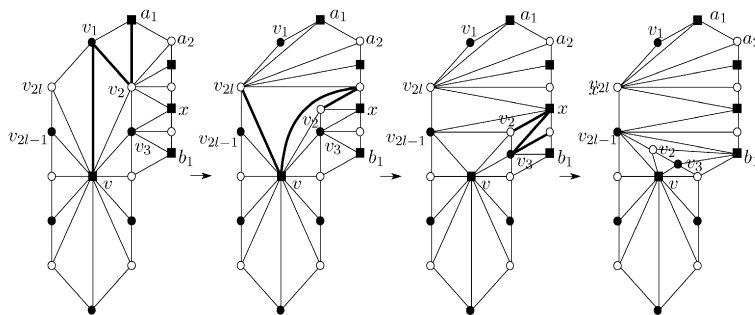
Suppose that G has a vertex v of degree at least 8 with link $a_3a_2a_1xb_1b_2b_3 \dots$. The edge elimination of the edge vx with respect to v is to identify a_i and b_i , for $i = 1, 2$, as shown in the right-hand side of Fig. 10. Similarly to the above two operations, we must preserve the simpleness of the graphs.

Lemma 9. Let G be a 3-chromatic triangulation on a closed surface F^2 .

- (i) Suppose that G has a red vertex v of degree 4 which is contractible at blue vertices and let G' be the resulting graph by the contraction. Then G is N -equivalent to $G' + r\gamma(1)$.
- (ii) Suppose that G has a contractible vertex v of degree 6, and let G' be the resulting graph by the contraction. Then G is N -equivalent to $G' + \Delta(1)$.

Proof. The statement (i) has already proved in [15], and hence we prove only (ii). Let $v_1v_2v_3v_4v_5v_6$ be the link of v in G , where $p = v_1$ and $q = v_4$. Let $a_1 \dots a_{2h-4}xv_3vv_1$ and $b_1 \dots b_{2k-4}xv_2vv_4$ be the links of v_2 and v_3 in G respectively, where $x, v \in N(v_2) \cap N(v_3)$. (See Fig. 11(1).)

Apply N -flips of $vv_1v_2a_1, va_2v_2a_3, \dots$ until v_2 has degree 4. (See Fig. 11(2).) Since v is contractible at $\{v_1, v_4\}$ in G , the link of v_2 does not contain v_6 in G , and hence the new edges $a_1v_6, a_2v_6, \dots, a_{2h-4}v_6$ and va_{2h-4} do not form multiple edges. Second, apply an N -flip of $v_6va_{2h-4}v_2$. (See Fig. 11(3).) Finally, apply N -flips of $vv_4v_3b_1, \dots$ until v_3 has degree 4. (See Fig. 11(4).) Then the resulting graph is nothing but $G' + \Delta(1)$. \square

Fig. 12. N -flips making a 2-subdividing pair.

Lemma 10. Let v be a red vertex of degree $2l \geq 8$ with link $v_1 v_2 \cdots v_{2l}$, and let G' be the graph obtained from G by an edge elimination of vv_i with respect to v . Then G is N -equivalent to $G' + by(1)$.

Proof. Let $v_3 v v_1 a_1 \dots a_p x$ and $v_2 v v_4 b_1 \dots b_q x$ be the links of v_2 and v_3 , respectively. Similarly to the proof of Lemma 9(ii), we may suppose that the link of v_2 does not contain v_{2l} . Note that v_2 and v_3 are blue and yellow, respectively. Applying N -flips to G similarly to the proof of Lemma 9(ii), we have the right-hand triangulation in Fig. 12, which has a 2-subdividing pair. Then the resulting graph is $G' + by(2)$. \square

Lemma 11. Let G be an even triangulation on a closed surface F^2 and let x and y be two neighboring vertices of degree 4. Let H be the graph obtained from G by a 2-elimination of xy . Then G is N -equivalent to $H + \Delta(1)$.

Proof. Let $abcy$ and $adcx$ be the links of x and y , respectively, and let R be the quadrilateral bounded by $abcd$. Let $bxydp_1 \dots p_{2k}$ be the link of a in G , where $k \geq 0$. Apply N -flips of $ydap_1$, $yp_2 ap_3, \dots, yp_{2k-2} ap_{2k-1}$, then we can obtain $H + \Delta(1)$. These operations do not break the simplicity of the graph because H is simple. \square

We call the three kinds of operations (a contraction of a vertex, a 2-elimination, an edge elimination and a removal of an octahedron) simply the *reductions*.

We can prove the following theorem, applying Lemmas 9, 10 and 11 inductively.

Lemma 12. Let G be a 3-chromatic triangulation on a closed surface F^2 . Suppose that G can be transformed into H by a sequence of reductions. Then G is N -equivalent to $H + rb(p) + by(q) + ry(r) + \Delta(m)$, where $|V_r(G)| - |V_r(H)| = p + r + m$, $|V_b(G)| - |V_b(H)| = p + q + m$ and $|V_y(G)| - |V_y(H)| = q + r + m$.

Note that in Lemma 12, we can take $m \in \{0, 1\}$, by Lemma 8. If we fix a possible value of m , we can take the numbers p, q and r uniquely.

5. Irreducible even triangulations

Let G be an even triangulation on a closed surface F^2 . We say that G is *irreducible* if any reduction for G results in a non-simple graph. For the sphere, the unique irreducible even triangulation is the octahedron [2].

In this section, we shall prove the following lemma.

Lemma 13. Every closed surface admits only finitely many irreducible even triangulations, up to homeomorphism.

We begin with the following lemma which describes the local structures of irreducible even triangulations. It is very important throughout this section and the key to prove the above lemma.

Lemma 14. *Let G be an irreducible even triangulation on a non-spherical closed surface. Then the following statements hold for G .*

- (i) G has no non-facial triangular region.
- (ii) If G has a vertex v of degree 4 with link $v_1v_2v_3v_4$, then there are two essential 3- or 4-cycles passing through $\{v, v_1, v_3\}$ and $\{v, v_2, v_4\}$, respectively.
- (iii) If G has a quadrangular region R of G , then R contains at most two inner vertices. If R has exactly two inner vertices, then they are adjacent.
- (iv) If G has a vertex v of degree $2k \geq 6$ with link $v_0 \cdots v_{2k-1}$, then for each i , either v_{i+1} and v_{i-1} are connected by a path P_i of length at most 2 such that $vv_{i+1}P_iv_{i-1}$ is an essential 3- or 4-cycle, or v_{i+2} and v_{i-2} are so.

Proof. (i) Suppose that G has a non-facial triangular region R with boundary ∂R . Take R to be innermost, that is, there is no such region in R . It is known that every planar graph has at least four vertices of degree less than 6, and hence there is an inner vertex v of degree 4 in R . Let $v_1v_2v_3v_4$ be the link of v in G . Since v is contractible at neither $\{v_2, v_4\}$ nor $\{v_1, v_3\}$, there are 3- or 4-cycles C_1 and C_2 passing through $\{v, v_1, v_3\}$ and $\{v, v_2, v_4\}$, respectively. Note that v_i and v_{i+2} do not lie on ∂R simultaneously. (For otherwise, v_iv_{i+2} bounds a smaller non-facial triangular region in R , a contradiction.) Therefore, both C_1 and C_2 are included in R , and hence they cross at v and another vertex, say x , in R . Since every 3-cycle in R other than ∂R bounds a face, we can find an octahedron with vertices $\{x, v, v_1, v_2, v_3, v_4\}$ in G . Clearly, this is removable from G , a contradiction.

(ii) Since v is contractible at neither $\{v_2, v_4\}$ nor $\{v_1, v_3\}$, there are 3- or 4-cycles C_1 and C_2 passing through $\{v, v_1, v_3\}$ and $\{v, v_2, v_4\}$, respectively. Similarly to the above case, if both C_1 and C_2 bound disks, then we can find a removable octahedron, a contradiction. Therefore, we may suppose that C_1 is essential. If C_2 bounds a 2-cell (say, contain v_1 in the interior), then C_1 and C_2 must cross at two vertices v and another vertex, say x , and two 3-cycles v_1xv_2 and v_1xv_4 bound faces of G , by (i). Since the 2-elimination of v and v_1 is impossible in G , we can find a 3- or 4-cycle, say C , through v_2, v and v_4 . Since C does not cross C_1 except v , C must be essential.

(iii) Let G' denote the subgraph of G consisting of the vertices and the edges on R and let $\partial G' = x_1x_2x_3x_4$ be the boundary cycle of G' . Since G' can be a subgraph of an even triangulation on the sphere, G' is 3-chromatic. Moreover, the maximal plane graph G'' obtained from G' by joining a new vertex u to x_1, x_2, x_3 and x_4 has minimum degree 4, by (i). Note that G'' has at least six vertices of degree less than 6, by Euler's formula, and every vertex of G'' except u, x_1, x_2, x_3, x_4 has even degree. Hence G'' has a vertex, say v , of degree 4, where $v \notin \{u, x_1, x_2, x_3, x_4\}$. Let $v_1v_2v_3v_4$ be the link of v in G' . By (ii), there are two essential 3- or 4-cycles passing through v , each of which cannot be contained in G' completely. Therefore, we may assume that $v_1 = x_1$ and $v_2 = x_2$. If $v_3, v_4 \notin V(\partial G')$, then we have $v_3x_3, v_4x_4 \in E(G')$, and hence the subgraph of G' induced by $\{v_1, v_2, v_3, v_4, x_3, x_4\}$ is not 3-colorable, contrary to $\chi(G') = 3$. Therefore, we may suppose that $x_3 \in V(\partial G')$. In this case, either $v_4 \in V(\partial G')$ or v_4 is adjacent to a vertex on $\partial G'$. Hence the lemma holds, since a triangular region of G' must be a face, by (i).

(iv) Since the edge elimination of the edge vv_i is impossible when $\deg(v) \geq 8$, and since the contraction of v is impossible when $\deg(v) = 6$, v_{i+1} and v_{i-1} are joined by a path P_i of length at most 2, or v_{i+2} and v_{i-2} are so. Since we can similarly deal with these two cases, we consider only the former. If we can choose P_i so that the 4-cycle $vv_{i+1}P_iv_{i-1}$ is essential, then we are done. Otherwise the 4-cycle $vv_{i+1}P_iv_{i-1}$ bounds a 2-cell, say D_i . By (iii), there are one or two vertices of degree 4 in D_i . By (ii), such a vertex of degree 4 is included in two essential 3- or 4-cycles. Hence there is an essential 3- or 4-cycle passing through v_{i-1}, v, v_{i+1} . \square

The following lemma plays an essential role to prove Lemma 13.

Lemma 15. (See Juwan, Malnič and Mohar [6].) *For any closed surface F^2 and any non-negative integer k , there exists a constant $f(k, F^2)$ such that if \mathcal{L} is a set of pairwise non-homotopic simple closed curves on F^2 such that any two elements of \mathcal{L} cross at most k times, then $|\mathcal{L}| \leq f(k, F^2)$.*

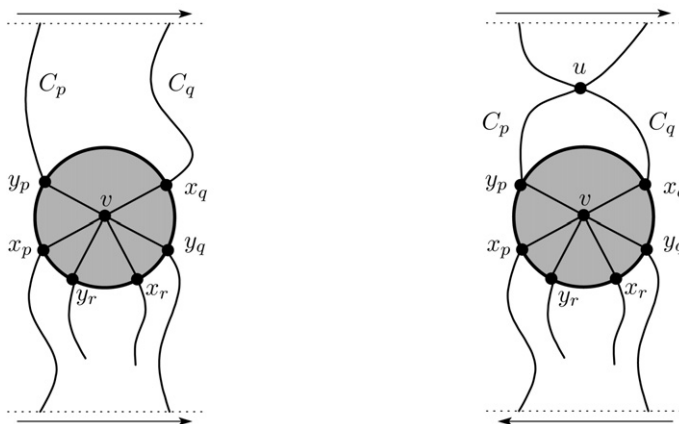


Fig. 13. The non-embeddability of \tilde{D} .

Note that for two simple closed curves l_1 and l_2 on a surface, “ l_1 crosses l_2 ” means that they cross transversely. On the other hand, “ l_1 intersects l_2 ” means that l_1 might touch l_2 and not necessarily cross l_2 .

We first bound the maximum degree $\Delta(G)$ of an irreducible even triangulation G .

Lemma 16. *Let G be an irreducible even triangulation on a non-spherical closed surface F^2 . Then the maximum degree of G is bounded by a constant depending only on F^2 .*

Proof. We shall prove that $\Delta(G) \leq 12f(1, F^2) + 5$, where $f(\cdot, F^2)$ is the function in Lemma 15. For contradictions, suppose that G has a vertex v with $\deg(v) \geq 12f(1, F^2) + 6$.

Let L_v be the link of v in G . Give a direction to L_v and denote the directed cycle by \vec{L}_v . Let $a_1, b_1, c_1, d_1, e_1, f_1, a_2, b_2, c_2, d_2, e_2, f_2, \dots, a_l, b_l, c_l, d_l, e_l, f_l$ be $6l$ consecutive vertices of L_v taken along \vec{L}_v , where $l \geq 2f(1, F^2) + 1$. Let $P(a, b)$ denote the path in L_v starting at $a \in V(L_v)$ and ending at $b \in V(L_v)$ along \vec{L}_v . By Lemma 14(iv), for each i , we can take a path P_i of length at most 2 joining b_i and d_i (or a_i and e_i) so that the 3- or 4-cycle, say C_i , obtained from P_i by joining v to the endpoints of P_i is essential. For a simple notation, we denote the endpoints of P_i by x_i and y_i , where either $(x_i, y_i) = (b_i, d_i)$ or $(x_i, y_i) = (a_i, e_i)$ for each i .

For some i , if C_i crosses C_j with $i \neq j$, then C_i has length 4 and the inner vertex of P_i (say z_i) coincides with some vertex of C_j other than v . Observe that if C_i and C_j cross transversely at least twice, then z_i coincides with either of x_j or y_j , and the inner vertex of P_j (say z_j) coincides with either of x_i and y_i . So we can find multiple edges, a contradiction. (Note that C_1 and C_2 do not share an edge connecting $\{x_i, y_i\}$ and $\{x_j, y_j\}$, since C_1 and C_2 cross twice.)

Since F^2 admits at most $f(1, F^2)$ simple closed curves which are pairwise non-homotopic and cross at most once, and since $l \geq 2f(1, F^2) + 1$, at least three, say C_p, C_q, C_r with $p < q < r$, of $\{C_1, \dots, C_l\}$ are homotopic on F^2 , by the Pigeonhole Principle. Let \tilde{D} be the configuration which is the union of the disk D bounded by L_v and the three cycles C_p, C_q and C_r of length 3 or 4, and consider whether \tilde{D} can be embedded in F^2 so that all of C_p, C_q and C_r are homotopic on F^2 .

Suppose that \tilde{D} has an embedding on F^2 such that C_p, C_q and C_r are 2-sided. Then no two of them cross, since any two homotopic 2-sided simple closed curves have an even number of crossings. Observe that in the embedding of \tilde{D} , C_p and C_q bound a *pinched annulus* A (i.e., an annulus where the two boundary components might touch several times), and that v is a pinched point of A . See the left-hand side of Fig. 13. Since x_p, y_p, x_q, y_q, x_r and y_r appear around v in this cyclic order, and since C_r does not cross both C_p and C_q , we cannot take C_r in A to be homotopic to C_p and C_q , a contradiction.

Suppose that \tilde{D} has an embedding on F^2 such that C_p, C_q and C_r are 1-sided. Then any two of them cross exactly once. Observe that C_p and C_q bound a pinched Möbius band M . Moreover, $x_p, y_p, x_q, y_q, x_r, y_r$ lie on L_v in this order, C_p and C_q touch at v and cross at some vertex u other than v . See the right-hand side of Fig. 13. Note that $u \notin \{x_p, y_p, x_q, y_q, x_r, y_r\}$. (For example, if $u = x_q$, then $y_p v u$ bounds a non-facial triangular region, since y_p and x_q are not adjacent in G . This contradicts Lemma 14(i).) Hence both x_r and y_r are inner vertices of the 2-cell region bounded by the 4-cycle $vy_q u x_p$, and hence there does not exist a 1-sided cycle of length at most 4 joining x_r and y_r through v , a contradiction. \square

Next we shall bound the diameter $\text{diam}(G)$ of an irreducible even triangulation G .

Lemma 17. *Let G be an irreducible even triangulation on a non-spherical closed surface F^2 . Then the diameter of G is bounded by a constant depending only on F^2 .*

Proof. We shall prove that $\text{diam}(G) \leq 35f(0, F^2) + 6$, where $f(\cdot, F^2)$ is the function in Lemma 15. For contradiction, suppose that G has two vertices x and y with distance at least $35f(0, F^2) + 7$. Let P be a path from x to y attaining that distance, and let $v_1 (=x), v_2, \dots, v_l$ be the vertices on P lying in this order, where $l \geq 5f(0, F^2) + 1$, so that v_i and v_{i+1} have distance exactly 7 on P , for $i = 1, \dots, l-1$. By Lemma 14(iv), for $i = 1, \dots, l$, we can take an essential 3- or 4-cycle passing through v_i , denoted by γ_i . Since v_i and v_j have distance at least 7 for any i and j , γ_i and γ_j are pairwise disjoint. Since F^2 admits only $f(0, F^2)$ pairwise non-crossing non-homotopic essential cycles, and since $l \geq 5f(0, F^2) + 1$, we can take six pairwise homotopic cycles in $\{\gamma_1, \dots, \gamma_l\}$, by the Pigeonhole Principle. Let D_1, \dots, D_6 be such six 3- or 4-cycles, each of which is 2-sided since they are pairwise homotopic and disjoint. We may suppose that D_1, \dots, D_6 lie on an annulus in this order. Let A be the annular subgraph of G bounded by D_1 and D_6 , and let A' be the subgraph of A bounded by D_2 and D_5 .

We shall prove that there is an inner vertex of degree 4 or 6 in A' . For getting contradictions, suppose that each inner vertex of A' has degree at least 8. Let T be the triangulation on the torus obtained from A' by pasting D_2 and D_5 . (If D_2 and D_5 have different length, say $3 = |D_2| < |D_5| = 4$, then subdivide an edge of D_2 , identify the two boundary components and add an edge to be a triangulation.) Let D be the cycle of T obtained from D_2 and D_5 by identifying them. Let $|V(D)| = t$ and $|V(T - D)| = s$. Since D_2 and D_5 can be assumed to be chordless in G , each vertex of D in T has degree at least 4. On the other hand, we have $s \geq 6$, since each of D_3 and D_4 has length at least 3. Therefore, since $3 \leq t \leq 4$ and $s \geq 6$, the average degree of T is

$$\geq \frac{4t + 8s}{t + s} \geq 6 + \frac{2(s - t)}{t + s} > 6.$$

This contradicts the fact that every triangulation on the torus has average degree exactly 6. Hence A' has an inner vertex v of degree at most 6.

We first suppose that $\deg(v) = 4$ and let $u_0 u_1 u_2 u_3$ be the link of v in G . By Lemma 14(ii), we can take two essential 3- or 4-cycles C_1 and C_2 through $\{v, u_0, u_2\}$ and $\{v, u_1, u_3\}$, respectively. Since C_1 and C_2 must be homotopic by the obstruction of D_1 and D_6 , both C_1 and C_2 have length exactly 4 and cross at exactly two vertices v and another vertex, say x . By Lemma 14(i), $xu_0 u_1$ and $xu_2 u_3$ bound faces in G . Since G is simple, we have $\deg(u_0) = k \geq 6$, and let $xu_1 v u_3 p_1 \dots p_{k-4}$ be the link of u_0 . (See the left-hand side of Fig. 14.) Now identify v and p_1 , and u_1 and p_2 , respectively, by the contraction of u when $\deg(v) = 6$, or the edge elimination otherwise. It must be possible since v and p_1 (and u_1 and p_2) cannot be joined by a path of length at most 2 not through v , by Lemma 14(i), (iii) and the obstruction of D_1 and D_6 . This contradicts that G is irreducible.

Second, suppose that there is no vertex of degree 4 in A' and there is v with $\deg(v) = 6$, and let $u_0 u_1 u_2 u_3 u_4 u_5$ be the link of v in G . Since v is not contractible, there are several paths P_0, \dots, P_{m-1} of length at most 2 each of which connects two vertices of distance 2 in the link of v . If we let C_i be the cycle obtained from P_i by joining v to the endpoints of P_i , for $i = 0, \dots, m-1$, then C_i is essential, by Lemma 14(iv). If there is a sequence $u_i, u_{i+1}, u_{i+2}, u_{i+3}$ such that P_a joins u_i and u_{i+2} and P_b joins u_{i+1} and u_{i+3} for some a, b , then the same argument as above follows.

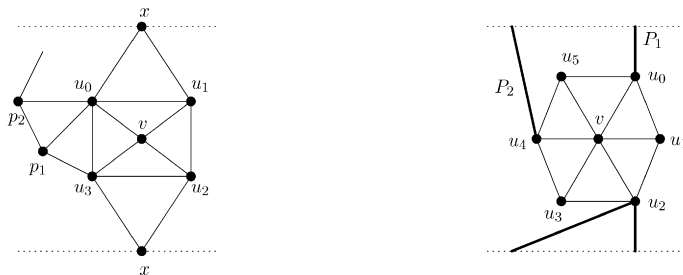


Fig. 14. The local structures around v in A .

Therefore, if P_1 is a path of length at most 2 joining u_0 and u_2 (possibly through u_3), then P_2 must join u_2 and u_4 . If both P_1 and P_2 have length 1, then u_5 has degree exactly 4, by Lemma 14(iii). Hence, we can get a contradiction around u_5 , as in the previous case. Therefore, we cannot forbid a contraction of v at $\{u_2, u_5\}$ without making multiple edges, and hence G is not irreducible, a contradiction. \square

Now we shall prove Lemma 13.

Proof of Lemma 13. Let G be a graph with maximum degree Δ and diameter d . Take any vertex v in G . Then the number of the neighbors of v in G is at most Δ , and that of the vertices of G whose distance is 2 from v is at most $\Delta(\Delta - 1)$. In general, the number of vertices of G with distance k from v is at most $\Delta(\Delta - 1)^{k-1}$. Therefore,

$$|V(G)| \leq 1 + \sum_{k=1}^d \Delta(\Delta - 1)^{k-1} = 1 + \frac{(\Delta - 1)^d - 1}{\Delta - 2}.$$

Hence, every irreducible even triangulation G on F^2 has a finite number of vertices, since its maximum degree and distance is bounded by a constant depending only on F^2 , by Lemmas 16 and 17. Thus, F^2 admits only finitely many irreducible even triangulations, up to homeomorphism. \square

6. Stably equivalence

Let G be an even triangulation on a closed surface F^2 . An even triangulation H is said to be a *refinement* of G if H satisfies the following two conditions:

- (i) H includes a subdivision of G .
- (ii) For a triangular face f of G bounded by $v_1 v_2 v_3$, let R_f be the plane subgraph of H corresponding to f , and let P_i be the path of the boundary cycle of R_f corresponding to the edge $v_i v_{i+1}$. Then each P_i has an odd length. Moreover, each inner vertex of P_i has an odd degree in R_f , and other vertices have even degree.

Observe that the condition (ii) guarantees an important property of H that if a 3-coloring c of the vertices v_1, v_2, v_3 of f in G is given, then R_f admits a 3-coloring c' such that $c(v_i) = c'(v_i)$ for $i = 1, 2, 3$. (This can be seen by the following procedures: Put R_f on the plane, put a 3-cycle $x_1 x_2 x_3$ surrounding R_f , and join x_i to all the vertices on P_i , for $i = 1, 2, 3$. Then we obtain an even triangulation on the plane, by the condition (ii). Consider a unique 3-coloring of this even triangulation.) Hence G and the refinement H have the same monodromy.

The following is a main result in this section, which is the most important argument in this paper controlled by the monodromies of two even triangulations.

Theorem 18. Let G and G' be two even triangulations on the same closed surface F^2 . Then G and G' have a common refinement if and only if their monodromies are congruent.

Let G be an even triangulation on a closed surface F^2 , and let σ_G be a monodromy of G . A simple closed curve l on F^2 is said to be *identity-assigned* if $\sigma_G([l]) = \text{id}$, where $[l]$ stands for the homotopy class of l . The following lemma is useful.

Lemma 19. (See [12,13].) *Let G be an even triangulation G on F^2 except the sphere and the projective plane. Then G admits an identity-assigned 2-sided essential closed curve l by σ_G . In particular, if l cannot be chosen to be non-separating, then l separates a crosscap from F^2 .*

Suppose that an even triangulation G has disjoint cycles C_1 and C_2 homotopic to an identity-assigned simple closed curve l . Let A be a submap of G bounded by C_1 and C_2 . Then A is 3-colorable since l is identity-assigned. We say that a cycle C of A is *bicolored* in G if the vertices of C are 2-colored in the unique 3-coloring of A . Cutting G along C is an operation transforming G into an even triangulation (if l is non-separating) or a disjoint union of two even triangulations (otherwise) with boundary components C^+ and C^- corresponding to C . Capping off is to paste 2-cells of C^+ and C^- , add two new vertices x^+ and x^- (called *auxiliary vertices*) to join them to all vertices of C^+ and C^- , respectively. Then the resulting map is an even triangulation since C is bicolored.

Let G be an even triangulation on the projective plane \mathbb{N}_1 . It is proved by Mohar [8] that every even triangulation on \mathbb{N}_1 has a face subdivision of some even embedding on \mathbb{N}_1 . So G is a face subdivision of some even embedding H . Cutting G along an essential cycle C of H , pasting a 2-cell to the boundary and putting a new vertex x joined to all vertices on the boundary, we obtain a 3-chromatic even triangulation. Similarly to the 2-sided case, we can define a *cutting*, a *capping off*, and an *auxiliary vertex* x .

In order to prove Theorem 18, we first prove the following. Let $\varepsilon(F^2)$ denote the Euler characteristic of a closed surface F^2 .

Lemma 20. *An even triangulation G on a non-spherical closed surface F^2 has a refinement D with pairwise disjoint 2-sided cycles A_1, \dots, A_p and 1-sided cycles B_1, \dots, B_q for some $p, q \geq 0$ with $2p + q \leq 2 - \varepsilon(F^2)$ such that the triangulation obtained from D by cutting and capping along A_1, \dots, A_p and B_1, \dots, B_q is 3-chromatic.*

Proof. If G is 3-chromatic, then G itself is a required refinement with no A_i and no B_j . Hence we suppose that G is not 3-chromatic, and find an identity-assigned essential cycle for cutting and capping off, until the resulting even triangulation by cutting and capping off is 3-chromatic. If F^2 is the projective plane, begin with Procedure 2. By Lemma 19, G admits an identity-assigned essential 2-sided simple closed curve l .

Procedure 1. Suppose that l is chosen to be non-separating. Let W be the closed face walk of G consisting of the faces through which l passes. We begin by coloring the three vertices of any fixed face f of W by 1, 2 and 3, and extend this to the vertices of each faces of W by the W -bijection. Since l is identity-assigned, there is a 3-coloring c of the vertices of W . (Note that c is just a 3-coloring of the vertices of each face in W , but might not be a 3-coloring of the vertices of W in G .) We put new vertices on all crossing points of l and edges of G . Let $v_1 \dots v_m$ be the cycle consisting of the vertices added to l . We use 1 and 2 to color v_1, \dots, v_m in this order by the following rule. Suppose that v_i is added on an edge xy of W . Observe that one of $c(x)$ and $c(y)$ is 1 or 2, say $c(x) = 1$. Then we color v_i by 1, and insert a vertex colored by $c(y)$ on the edge $v_i x$. Do the same procedure to v_1, \dots, v_m . If v_i and v_{i+1} have the same color, then insert a vertex colored by another color (1 or 2) to the edge $v_i v_{i+1}$. At the end of this step, add edges and vertices so that the resulting graph, denoted by K , is an even triangulation and has a 3-coloring in W . Then K is a refinement of G . Note that the cycle, denoted by C , corresponding to l is a bicolored cycle of K . Here observe that adding several 2-subdividing pairs of vertices to an edge of C , we can increase the length of C to any even number. Now we cut open K along C , where C^+ and C^- denote the two boundary cycles corresponding to C . Note that $|C^+| = |C^-|$. Let H be the even triangulation obtained from K by cutting and capping along C by auxiliary vertices x^+ and x^- , respectively. Clearly, H is an even triangulation on the surface whose Euler characteristic is increased by 2.

Procedure 2. Suppose that l is separating. By the same procedures as in Case 1, we can obtain a refinement K which has a bicolored cycle L separating a crosscap from F^2 . Let P be the even triangulation on \mathbb{N}_1 with an auxiliary vertex v . Here we may suppose that P is not 3-chromatic (since we are finding a cycle whose cutting makes a triangulation 3-colorable). As mentioned above, P is a face subdivision of some even embedding Q . Since Q is non-bipartite, we can find an essential odd cycle C in Q . Here note that the length of C can be increased to any odd number by inserting 2-subdividing pairs to an edge of C . Moreover, C can be chosen not to pass through v . (For otherwise, i.e., if every essential cycle of Q passed through v , then $Q - v$ would be a planar embedding in \mathbb{N}_1 , but this contradicts the 2-representativity of Q .) Let H be the even triangulation obtained from K by cutting and capping along such a 1-sided cycle C by an auxiliary vertex x . Note that H is an even triangulation on the surface whose Euler characteristic increased by 1.

Applying Procedures 1 and 2 repeatedly, we get a 3-chromatic even triangulation \tilde{D} on a closed surface F'^2 . However, in the process of Procedures 1 and 2, we should take l not to cross edges incident to auxiliary vertices. It is possible since two links of two distinct auxiliary vertices are disjoint. Let D be the even triangulation on F^2 obtained from \tilde{D} by removing all auxiliary vertices and identifying the corresponding boundaries. Then D is a refinement of G .

Suppose that we apply Procedures 1 and 2, p and q times, respectively. Let A_1, \dots, A_p be the disjoint 2-sided cycles in D arisen in Procedure 1, and let B_1, \dots, B_q be the disjoint 1-sided cycles in D arisen in Procedure 2. Since $\varepsilon(F'^2) \leq 2$, we have $2a + b \geq \varepsilon(F'^2) - \varepsilon(F^2) \geq 2 - \varepsilon(F^2)$. Therefore, D is a required refinement of G . \square

Proof of Theorem 18. Let G and G' be two even triangulations on the same surface F^2 with the same monodromy. By Lemma 20, G has a refinement D with identity-assigned p essential 2-sided cycles A_1, \dots, A_p and q 1-sided cycles B_1, \dots, B_q . Since G' has the same monodromy as G , G' has a refinement D' with identity-assigned p essential 2-sided cycles A'_1, \dots, A'_p and q 1-sided cycles B'_1, \dots, B'_q , where we may suppose that A_i and A'_i are homotopic on F^2 , and so are B_j and B'_j , for each i, j , and that $|A_i| = |A'_i|$ and $|B_j| = |B'_j|$ for each i, j since the length of the cycles can be increased to any odd (respectively even) number by inserting 2-subdividing pairs in Procedure 1 (respectively 2).

Let \tilde{D} be the 3-chromatic triangulation obtained from G by cutting along A_i and B_j for each i and j . The surface where \tilde{D} embeds, denoted by Σ , has $2a + b$ boundary components, and each boundary cycle is 2-colored. Let A_i^+ and A_i^- be the two boundary components of \tilde{D} corresponding to A_i for $i = 1, \dots, p$, and let $B_j^\pm = B_j^+ \cup B_j^-$ be the boundary component of \tilde{D} corresponding to B_j , for $j = 1, \dots, q$. Similarly, we define $A_i'^+, A_i'^-, B_j'^\pm$ for D' . Suppose that \tilde{D} and \tilde{D}' are 3-colored by $\{1, 2, 3\}$.

Embed \tilde{D} and \tilde{D}' to Σ simultaneously so that the following is satisfied:

- (i) A_i^+ and $A_i'^+$ are identified so that the same colors coincide, for $i = 1, \dots, p$.
- (ii) The identification (i) determines a bijection of the vertices of A_i in D and those of A'_i in D' for each i . By this bijection, we identify A_i^- and $A_i'^-$, for $i = 1, \dots, p$.
- (iii) B_j^\pm and $B_j'^\pm$ are identified so that the same colors coincide, for $j = 1, \dots, q$.
- (iv) Any other intersection of \tilde{D} and \tilde{D}' is the crossing of two edges of \tilde{D} and \tilde{D}' at their inner point.

It is a very important observation that the identification (ii) must be color-preserving, since the monodromies of G and G' coincide.

Put a new vertex v on each of the crossing of $e \in E(\tilde{D})$ and $e' \in E(\tilde{D}')$ arisen in (iv). Since \tilde{D} and \tilde{D}' are 3-colored, there is a color appeared in both an endpoint of e and that of e' . We color v by this color. Do the same for all intersections arisen in (iv). Then we can see that if two end points of e are colored by 1, 2 in \tilde{D} , then all vertices on the path of the resulting graph corresponding to e are colored by 1, 2. In this path, if two vertices with the same color are consecutive, then we insert a vertex with another color into the edge. Do the same for all edges. Let \tilde{X} be the resulting triangulation

on Σ obtained from it by adding edges and vertices suitably not breaking the 3-colorability. Then \tilde{X} is a common refinement of \tilde{D} and \tilde{D}' .

Let X be the even triangulation on F^2 obtained from \tilde{X} by identifying A_i^+, A_i^- , and B_j^+, B_j^- for each i and j . Remember that D (respectively D') is a refinement of G (respectively G'), and that \tilde{D} (respectively \tilde{D}') is obtained from D (respectively D') by cutting. Hence X is a common refinement of G and G' . \square

Let G and G' be two even triangulations on the same closed surface. We say that G and G' are *stably equivalent* if G with some 2-subdividing pairs and octahedra added and G' with some 2-subdividing pairs and octahedra added are N -equivalent. In particular, in the case when G and G' are 3-chromatic, they are stably equivalent if there exist integers p, q, r, m and p', q', r', m' such that

$$G + rb(p) + by(q) + ry(r) + \Delta(m) \approx_N G' + rb(p') + by(q') + ry(r') + \Delta(m').$$

Note that G and G' may not have the same number of vertices.

Lemma 21. *Let G be an even triangulation on a closed surface F^2 and let H be a refinement of G . Then G and H are stably equivalent.*

Proof. For each edge e of G , we can find a corresponding path \hat{e} of odd length in H . We use induction on $\sum_{e \in E(G)} |\hat{e}|$ and $|V(H)|$, where $|\hat{e}|$ denotes the length of the path \hat{e} . If $|V(H)| = |V(G)|$, then we have $H = G$, and hence we have nothing to do. If $\sum_{e \in E(G)} |\hat{e}| = |E(G)|$, then H has G as its subgraph. By Lemma 14(i), the 2-cell region of H corresponding to a face of G can be transformed into just a face by the reductions. Lemma 12 guarantees the stable equivalence of G and H , and the first step of induction is verified.

Let $P = r_1 b_1 r_2 b_2 \cdots r_k b_k$ be the path of H corresponding to some edge e of G , where $k \geq 2$, where r_1 and b_k are vertices of G . Let R_1 and R_2 be the 2-cell regions of H corresponding to the two triangular faces of G incident to e . In particular, let ∂R_i denote the boundary cycle of R_i and let $\text{Int } R_i = R_i - \partial R_i$, for $i = 1, 2$. By the method to take a refinement of G , we can color the vertices in $R_1 \cup R_2$ by red, blue and yellow so that each path of ∂R_i corresponding to the edge of G is colored by two colors. It makes our argument clearly understandable. In this proof (even in the former part), regard the vertices denoted by r, b and y with subscripts as those colored by red, blue and yellow, respectively.

We shall prove that P can be shortened by the reductions. We first suppose that H has a 4-cycle $C = r_1 v_1 r_2 v_2$ bounding a 2-cell region D for some vertices v_1 and v_2 belonging to R_1 and R_2 , respectively. We suppose that C is *outermost*, that is, there is no such 4-cycle through r_1 and r_2 containing C in the interior. If v_1 and v_2 have the same color, we may assume that D has just one inner vertex, by Lemma 14(iii) (for otherwise, apply a reduction to get a smaller H). Otherwise, we may suppose that v_1 and b_1 have the same color. We apply Lemma 14(iii) to two quadrangular regions $r_1 v_1 r_2 b_1$ and $r_1 v_2 r_2 b_1$, and then D is assumed to have exactly two vertices. In the former case, apply the contraction of the unique vertex of degree 4, and in the latter case, apply the 2-elimination of the edge. Both reductions are possible because C is outermost in H . Then the length of P can be shortened. Let K be the resulting graph by the reduction. By Lemma 12, since H is contractible to K by reductions, H is N -equivalent to K with a 2-subdividing pair or an octahedron. By the induction hypothesis, K and G are stably equivalent, and hence so are H and G .

Second, we consider the case when R_1 has a vertex v_1 adjacent to both r_1 and r_2 , but R_2 does not. Choose v_1 so that the quadrangular region bounded by $r_1 b_1 r_2 v_1$ is maximal. This region is assumed to contain one vertex or just one diagonal $v_1 b_1$, depending on whether or not v_1 and b_1 have the same color. In the former case, apply the contraction of the vertex of degree 4 at $\{v_1, b_1\}$ in H . Then the resulting graph is simple because r_1 and r_2 have no common vertex other than v_1 and b_1 . Hence we can do similarly to the first case. We deal with the latter case in the following case.

Finally, we suppose that either there is no common neighbor of r_1 and r_2 other than b_1 . Let $r_1 y_1 r'_1 y_2 r'_2 \cdots r'_{l-1} y_l r_2$ be the path on the link of b_1 lying on R_1 , where $l \geq 1$. Add a 2-subdividing pair to each $b_1 r'_i$, for $i = 1, \dots, l-1$, and next add a 2-subdividing pair to each $b_1 y_i$, for $i = 1, \dots, l$.

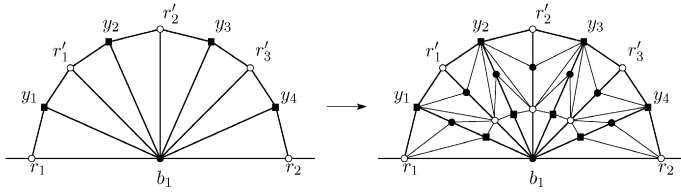


Fig. 15. Add 2-subdividing pairs.

(See Fig. 15.) If we let H' be the resulting graph, then $H' \approx_N H + rb(l-1) + by(l)$. (In fact, H' might not be 3-chromatic. However it is locally colored by three colors and it helps our argument precisely.) Now apply the 2-eliminations at the 2-subdividing pairs added on all b_1y_i 's simultaneously. Observe that by the assumption of the current case, the resulting graph (denoted by H'') has no multiple edges and loops. Clearly, we have $H' \approx_N H'' + \Delta(l)$, by Lemma 9(ii). Since the length of P is shortened in H'' , we can apply induction hypothesis to H'' and hence H'' and G are stably equivalent and so are H' and G . Therefore, G and H are stably equivalent. \square

Lemma 22. Any two even triangulations on the same closed surface which have congruent monodromies are stably equivalent.

Proof. Any two even triangulations G and G' which have congruent monodromies have a common refinement H , by Theorem 18. Hence G and G' are stably equivalent via H , by Lemma 21. \square

7. Proof of the theorems

In this section, we shall prove our main theorems. At first, we prove Theorem 3. Before it, we shall prepare the following lemma which describes the relation of three integers satisfying triangle inequality.

Lemma 23. Three non-negative integers m, n and l satisfy triangle inequality $|m - n| \leq l \leq m + n$ if and only if

$$m = \alpha + \beta + \omega, \quad n = \beta + \gamma + \omega, \quad l = \alpha + \gamma + \omega$$

for some integers $\alpha, \beta, \gamma \geq 0$ and $\omega \in \{0, 1\}$.

Proof. If $m = \alpha + \beta + \omega$, $n = \beta + \gamma + \omega$ and $l = \alpha + \gamma + \omega$, then we have $|m - n| \leq l \leq m + n$, and hence the sufficiency is obvious. So we prove the necessity by induction on $m + n + l$. If $m + n + l = 0$, then we have $\alpha = \beta = \gamma = \omega = 0$, and hence we get the first step of induction. Suppose that m, n and l satisfy triangle inequality $|m - n| \leq l \leq m + n$, where we suppose that $n \leq m \leq l$. Let $m' = m - 1$ and $l' = l - 1$.

If $m - n \geq 1$ or $l \geq 2$, then $|m' - n| \leq l' \leq m' + n$. Therefore, by induction hypothesis, we have $m' = \alpha' + \beta' + \omega'$, $n = \beta' + \gamma' + \omega'$, $l' = \alpha' + \gamma' + \omega'$ for some integers $\alpha', \beta', \gamma' \geq 0$ and $\omega' \in \{0, 1\}$. Hence we have $\alpha = \alpha' + 1 \geq 1$, $\beta = \beta' \geq 0$, $\gamma = \gamma' \geq 0$, $\omega = \omega' \in \{0, 1\}$.

Otherwise, i.e., $m = n$ and $l \leq 1$, then we have either $(m, n, l) = (0, 0, 1)$ or $(m, n, l) = (1, 1, 1)$. The former does not satisfy triangle inequality, and in the latter case, we can take $\alpha = \beta = \gamma = 0$ and $\omega = 1$. \square

Proof of Theorem 3. By Lemma 13, every closed surface F^2 admits only finitely many irreducible 3-chromatic triangulations, up to homeomorphism. We denote them by I_1, \dots, I_h . For each I_i with tripartition $V(I_i) = R \cup B \cup Y$ and for each permutation σ of $\{R, B, Y\}$, let I_i^σ be a copy of I_i with $V_r(I_i^\sigma) = \sigma(R)$, $V_b(I_i^\sigma) = \sigma(B)$ and $V_y(I_i^\sigma) = \sigma(Y)$. Let $\{J_1, \dots, J_n\}$ be the union of the six graphs for I_i for $i = 1, \dots, h$.

By Lemma 22, since n is finite, there exists a 3-tuple (N_r, N_b, N_y) of non-negative integers such that for all i, j with $1 \leq i < j \leq n$,

$$J_i + rb(p_i) + by(q_i) + ry(r_i) + \Delta(m_i) \approx_N J_j + rb(p_j) + by(q_j) + ry(r_j) + \Delta(m_j) \quad (1)$$

where $m_i, m_j \in \{0, 1\}$, by Lemma 8, and

$$\begin{aligned} N_r &= |V_r(J_i)| + p_i + r_i + m_i = |V_r(J_j)| + p_j + r_j + m_j, \\ N_b &= |V_b(J_i)| + p_i + q_i + m_i = |V_b(J_j)| + p_j + q_j + m_j, \\ N_y &= |V_y(J_i)| + q_i + r_i + m_i = |V_y(J_j)| + q_j + r_j + m_j. \end{aligned}$$

Let G and G' be two 3-chromatic triangulations on F^2 in the theorem. Then, by Lemma 23, there exist integers $\alpha, \beta, \gamma \geq 0$ and $\omega \in \{0, 1\}$ such that

$$\begin{aligned} |V_r(G)| &= |V_r(G')| = N_r + \alpha + \beta + \omega, \\ |V_b(G)| &= |V_b(G')| = N_b + \beta + \gamma + \omega, \\ |V_y(G)| &= |V_y(G')| = N_y + \alpha + \gamma + \omega. \end{aligned}$$

Since G is contractible to an irreducible even triangulation, say J_k , and hence, by Lemma 12, we have

$$G \approx_N J_k + rb(p_k + \beta) + by(q_k + \gamma) + ry(r_k + \alpha) + \Delta(m_k + \omega).$$

Since $J_k + rb(p_k) + by(q_k) + ry(r_k) + \Delta(m_k) \approx_N J_1 + rb(p_1) + by(q_1) + ry(r_1) + \Delta(m_1)$ by (1) and since a 2-subdividing pair and an octahedron added can be moved to anywhere, by Lemmas 6 and 7, we have

$$G \approx_N J_1 + rb(p_1 + \beta) + by(q_1 + \gamma) + ry(r_1 + \alpha) + \Delta(m_1 + \omega).$$

By the same argument, since G' has the same tripartition size as G , we have

$$G' \approx_N J_1 + rb(p_1 + \beta) + by(q_1 + \gamma) + ry(r_1 + \alpha) + \Delta(m_1 + \omega).$$

Therefore, the theorem follows. \square

Let G be an even triangulation on a closed surface F^2 . When G can be transformed into G' by a sequence of N -flips and P_2 -flips, we denote $G \sim_N G'$. By Lemma 6, using P_2 -flips in addition, we can move a 2-subdividing pair to any edge. Therefore, by Lemma 7, we can define $G + \Gamma(p) + \Delta(m)$ to be an even triangulation obtained from G by adding p 2-subdividing pairs to some edge e of G and m octahedra to some face. This expression does not depend on the edge and the face where the 2-subdividing pairs and the octahedra are added.

Proof of Theorem 2. By the N -equivalence (2), (3) and (4) in the previous proofs, there exists an integer N' such that for all i and j ,

$$I_i + \Gamma(p_i) + \Delta(m_i) \sim_N I_j + \Gamma(p_j) + \Delta(m_j),$$

where $m_i \in \{0, 1\}$ and $N' = |V(I_i)| + 2n_i + 3m_i$ for $i = 1, \dots, n$.

Let $N = N' + 3$ and let G and G' be even triangulations on F^2 with $|V(G)| = |V(G')| \geq N$ which have a congruent monodromy. Since G is contractible to some I_k , we have $G \sim_N I_k + \Gamma(p') + \Delta(m')$ for some $p' \geq 0$ and $m' \in \{0, 1, 2\}$ such that $2p' + 3m' = |V(G)| - |V(I_k)|$. By Lemma 8, we may suppose that $0 \leq m' - m_k \leq 1$. (For otherwise, we have $0 = m' < m_k = 1$. Since $|V(G)| \geq N = N' + 3$, we have $p' - p_k \geq 3$ and hence $G \sim_N I_k + \Gamma(p') + \Delta(0) \sim_N I_k + \Gamma(p' - 3) + \Delta(2)$.) Moreover, since $N = N' + 3$ again, we have $p' \geq p_k$. Therefore,

$$\begin{aligned} G &\sim_N I_k + \Gamma(p') + \Delta(m') \\ &\sim_N I_k + \Gamma(p_k + a) + \Delta(m_k + b) \\ &\sim_N I_1 + \Gamma(p_1 + a) + \Delta(m_1 + b), \end{aligned}$$

where $a = p' - p_k \geq 0$ and $b = m' - m_k \geq 0$. We can do the same for G' since $|V(G)| = |V(G')|$. Thus, the theorem follows. \square

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